

A priori L^∞ -estimates for degenerate complex Monge-Ampère equations

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Abstract : We study families of complex Monge-Ampère equations, focusing on the case where the cohomology classes degenerate to a non big class. We establish uniform a priori L^∞ -estimates for the normalized solutions, generalizing the recent work of S. Kolodziej and G. Tian. This has interesting consequences in the study of the Kähler-Ricci flow.

1 Introduction

Let $\pi : X \longrightarrow Y$ be a non degenerate holomorphic mapping between compact Kähler manifolds such that $n := \dim_{\mathbb{C}} X \geq m := \dim_{\mathbb{C}} Y$. Let ω_X, ω_Y Kähler forms on X and Y respectively. Let $F : X \longrightarrow \mathbb{R}^+$ be a non negative function such that $F \in L^p(X)$ for some $p > 1$.

Set $\omega_t := \pi^*(\omega_Y) + t\omega_X$, $t > 0$. We consider the following family of complex Monge-Ampère equations

$$(\star)_t \quad \begin{cases} (\omega_t + dd^c \varphi_t)^n &= c_t t^{n-m} F \omega_X^n \\ \max_X \varphi_t = 0 &= 0 \end{cases}$$

where φ_t is ω_t -plurisubharmonic on X and $c_t > 0$ is a constant given by

$$c_t t^{n-m} \int_X F \omega_X^n = \int_X \omega_t^n.$$

It follows from the seminal work of S.T. Yau [Y] and S. Kolodziej [K 1], [K 2] that the equation $(\star)_t$ admits a unique continuous solution. (Observe that for $t \in]0, 1]$, ω_t is a Kähler form).

Our aim here is to understand what happens when $t \rightarrow 0^+$, motivated by recent geometrical developpments [ST], [KT]. When $n = m$, the cohomology class ω_0 is big and semi-ample and this problem has been adressed by several authors recently (see [CN], [EGZ], [TZ], [To]).

We focus here on the case $m < n$. This situation is motivated by the study of the Kähler-Ricci flow on manifolds X of intermediate Kodaira dimension $1 \leq \text{cod}(X) \leq n - 1$. When $n = 2$ this has been studied by J.Song and G.Tian [ST].

In a very recent and interesting paper [KT], S. Kolodziej and G. Tian were able to show, under a technical geometric assumption on the fibration π , that the solutions (φ_t) are uniformly bounded on X when $t \searrow 0^+$.

The purpose of this note is to (re)prove this result without any technical assumption and with a different method: we actually follow the strategy introduced by S. Kolodziej in [K] and further developed in [EGZ], [BGZ].

THEOREM. *There exists a uniform constant $M = M(\pi, \|F\|_p) > 0$ such that the solutions to the Monge-Ampère equations $(\star)_t$ satisfy*

$$\|\varphi_t\|_{L^\infty(X)} \leq M, \quad \forall t \in]0, 1].$$

It follows from our result that Theorems 1 and 2 in [KT] hold without any technical assumption on the fibration (see condition 0.2 in [KT]).

This result has been announced by J-P. Demailly and N. Pali [DP].

2 Proof of the theorem

2.1 Preliminary remarks

Uniform control of c_t . Observe that $\omega_0^k = 0$ for $m < k \leq n$, hence for all $t \in]0, 1]$,

$$\omega_t^n = \sum_{k=1}^m \binom{n}{k} t^{n-k} \omega_0^k \wedge \omega_X^{n-k}.$$

Note that $]0, 1] \ni t \mapsto t^{m-n} \omega_t^n$ is increasing (hence decreases as $t \searrow 0^+$) and satisfies for $t \in]0, 1]$

$$(1) \quad \binom{n}{m} \frac{\omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \leq \frac{\omega_t^n}{t^{n-m} \int_X \omega_0^m \wedge \omega_X^{n-m}} \leq \frac{\omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}}.$$

In particular $t \mapsto c_t$ is increasing in $t \in]0, 1]$ and

$$0 < \binom{n}{m} \frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X F \omega_X^n} =: c_0 \leq c_t \leq c_1.$$

Uniform control of densities. Let J_π denote the (modulus square) of the Jacobian of the mapping π , defined through

$$\omega_0^m \wedge \omega_X^{n-m} = J_\pi \omega_X^n.$$

Let us rewrite the equation $(\star)_t$ as follows

$$(\omega_t + dd^c \varphi_t)^n = f_t \omega_t^n,$$

where for $t \in]0, 1]$

$$0 \leq f_t := c_t t^{n-m} F \frac{\omega_X^n}{\omega_t^n} \leq c_1 \frac{F}{J_\pi}.$$

Observe that

$$\int_X f_t \omega_t^n = c_t t^{n-m} \int_X F \omega_t^n = \int_X \omega_t^n =: Vol_{\omega_t}(X),$$

hence (f_t) is uniformly bounded in $L^1(\omega_t/V_t)$, $V_t := Vol_{\omega_t}(X)$. We actually need a slightly stronger information.

Lemma 2.1 *There exists $p' > 1$ and a constant $C = C(\pi, \|F\|_{L^p(X)}) > 0$ such that for all $t \in]0, 1]$*

$$\int_X f_t^{p'} \omega_t^n \leq C Vol_{\omega_t}(X).$$

Proof of the lemma. Set $V_t := Vol_{\omega_t} = \int_X \omega_t^n$ and observe that

$$0 \leq f_t \frac{\omega_t^n}{V_t} \leq c_1 F \frac{\omega_X^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} = C_2 F \omega_X^n,$$

where $C_2 := c_1 \int_X J_\pi \omega_X^n$.

This shows that the densities f_t are uniformly in L^1 w.r.t. the normalized volume forms ω_t^n/V_t .

Since J_π is locally given as the square of the modulus of a holomorphic function which does not vanish identically, there exists $\alpha \in]0, 1[$ such that $J_\pi^{-\alpha} \in L^1(X)$. Fix $\beta \in]0, \alpha[$ satisfying the condition $\beta/p + \beta/\alpha = 1$. It follows from Hölder's inequality that

$$\int_X f_t^\beta \omega_X^n \leq \left(\int_X F^p \omega_X^n \right)^{\beta/p} \left(\int_X J_\pi^{-\alpha} \omega_X^n \right)^{\beta/\alpha}.$$

Setting $\varepsilon := \beta/q$ and using Hölder's inequality again, we obtain

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \int_X f_t^\varepsilon F \omega_X^n.$$

Now applying again Hölder inequality we get

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \left(\int_X f_t^\beta \omega_X^n \right)^{1/q} \|F\|_{L^p(X)}.$$

Therefore denoting by $p' := 1 + \varepsilon$, we have the following uniform estimate

$$\int_X f_t^{p'} \frac{\omega_t^n}{V_t} \leq C(\pi, \|F\|_{L^p(X)}), \forall t \in]0, 1],$$

where

$$C(\pi, \|F\|_{L^p(X)}) := C_2 \left(\int_X J_\pi^{-\alpha} \omega_X^n \right)^{\beta/\alpha q} \|F\|_{L^p(X)}^{1+\beta/q}.$$

►

2.2 Uniform domination by capacity

We now show that the measure $\mu_t := f_t \omega_t^n / \text{Vol}_{\omega_t}$ are uniformly strongly dominated by the normalized capacity $\text{Cap}_{\omega_t} / \text{Vol}_{\omega_t}(X)$. It actually follows from a carefull reading of the no parameter proof given in [EGZ], [BGZ].

Lemma 2.2 *There exists a constant $C_0 = C_0(\pi, \|F\|_{L^p(\omega_X^n)}) > 0$ such that for any compact set $K \subset X$ and $t \in]0, 1]$,*

$$\mu_t(K) \leq C_0^n \left(\frac{\text{Cap}_{\omega_t}(K)}{\text{Vol}_{\omega_t}(X)} \right)^2.$$

Proof: Fix a compact set $K \subset X$. Set $V_t := \text{Vol}_{\omega_t}(X)$. Hölder's inequality yields

$$\mu_t(K) \leq \left(\int_X f_t^{p'} \frac{\omega_t^n}{V_t} \right)^{1/p'} \left(\int_K \frac{\omega_t^n}{V_t} \right)^{1/q'}.$$

It remains to dominate uniformly the normalized volume forms ω_t^n / V_t by the normalized capacities $\text{Cap}_{\omega_t} / V_t$. Fix $\sigma > 0$ and observe that for any $t \in]0, 1]$,

$$\int_K \frac{\omega_t^n}{V_t} \leq \int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} T_{\omega_t}(K)^\sigma,$$

where

$$V_{K,\omega_t} := \sup\{\psi \in \text{PSH}(X, \omega_t); \psi \leq 0, \text{ on } K\}$$

is the ω_t -extremal function of K and $T_{\omega_t}(K) := \exp(-\sup_X V_{K,\omega_t})$ is the associated ω_t -capacity of K (see [GZ 1] for their properties).

Observe that $\omega_t^n / V_t \leq c_1 \omega_1^n$ and $\omega_t \leq \omega_1$, hence the family of functions $V_{K,\omega_t} - \max_X V_{K,\omega_t}$ is a normalized family of ω_1 -psh functions. Thus there exists $\sigma > 0$ which depends only on (X, ω_1) and a constant $B = B(\sigma, X, \omega_1)$ such that ([Z])

$$\int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} \leq B, \forall t \in]0, 1].$$

The Alexander-Taylor comparison theorem (see Theorem 7.1 in [GZ 1]) now yields for a constant $C_3 = C_3(\pi, \|F\|_{L^p(X)})$

$$\mu_t(K) \leq C_3 \exp \left[-\sigma \left(\frac{V_t}{\text{Cap}_{\omega_t}(K)} \right)^{1/n} \right], \forall t \in]0, 1].$$

We infer that there is a constant $C_4 = C_4(\pi, \|F\|_{L^p(X)})$ such that

$$(2) \quad \mu_t(K) \leq C_4 \left(\frac{\text{Cap}_{\omega_t}(K)}{V_t} \right)^2, \forall t \in]0, 1].$$

2.3 Uniform normalization

The comparison principle (see [K], [EGZ]) yields for any $s > 0$ and $\tau \in [0, 1]$

$$\tau^n \frac{\text{Cap}_{\omega_t}(\{\varphi_t \leq -s - \tau\})}{V_t} \leq \int_{\{\varphi_t \leq -s\}} \frac{(\omega_t + dd^c \varphi_t)^n}{V_t}.$$

It is now an exercise to derive from this inequality an a priori L^∞ -estimate,

$$\|\varphi_t\|_{L^\infty(X)} \leq C_5 + s_0(\omega_t),$$

where $s_0(\omega_t)$ (see [EGZ], [BGZ]) is the smallest number $s > 0$ satisfying the condition $e^n C_0^n \text{Cap}_{\omega_t}(\{\psi \leq -s\})/V_t \leq 1$ for all $\psi \in PSH(X, \omega_t)$ such that $\sup_X \psi = 0$. Recall from ([GZ 1], Prop. 3.6) that

$$\frac{\text{Cap}_{\omega_t}(\{\psi \leq -s - \tau\})}{V_t} \leq \frac{1}{s} \left(\int_X (-\psi) \frac{\omega_t^n}{V_t} + n \right).$$

Since $\frac{\omega_t^n}{V_t} \leq C_1 \omega_1^n$, it follows that

$$\frac{\text{Cap}_{\omega_t}(\{\psi \leq -s - \tau\})}{V_t} \leq \frac{1}{s} \left(C_1 \int_X (-\psi) \omega_1^n + n \right).$$

Since ψ is ω_1 -psh and normalized, we know that there is a constant $A = A(X, \omega_1) > 0$ such that $C_1 \int_X (-\psi) \omega_1^n \leq A$ for any such ψ . Therefore $s_0(\omega_t) \leq s_0 := e^n C_0^n (A + n)$ for any $t \in]0, 1]$. Finally we obtain the required uniform estimate for all $t \in]0, 1]$.

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